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BIOGRAPHY.

DE MORGAN.

BY GEORGE BRUCE HALSTED.

AUGUSTUS DE MORGAN is fortunately a well-known character. The world has an excellent sketch of him by his pupil Jevons in the Encyclopædia Britannica, from which I copy literally some of the preliminary biographic data here given. If I am able to add anything specifically new to this part of the paper, it is from a life-long study of his works, and many conversations about him with the great Sylvester, who knew him very intimately.

Augustus De Morgan was born in June, 1806, in India. On his mother's side he was descended from James Dodson, F. R. S., author of the Anti-Logarithmic Canon and other mathematical works of merit, and a friend of De Moivre. Our Augustus lost one eye in his early infancy, and this prevented his joining in the usual games. He read algebra "like a novel," and pricked out equations on the school pew instead of listening to the sermons.

When 16 years old he entered Trinity College, Cambridge, and studied his mathematics partly under the tuition of Airy, making many friends, including his teachers Whewell and Peacock. In 1825 he gained a Trinity Scholarship.

But De Morgan's attention was by no means confined to mathematics, and his love of wide reading somewhat interfered with his success in the mathematical tripos, in which he took fourth place in 1827, before he had completed his 21st year.

He was prevented from taking his M. A. degree, or from obtaining a fel-

lowship, by his conscientious objection to signing the theological tests then required from masters of arts and Fellows at Cambridge. Jevons says, "a strong repugnance to any sectarian restraint upon the freedom of opinion was one of De Morgan's most marked characteristics throughout life."

A career in his own university being thus closed against him, on the twenty-third of February, 1828, he was elected Professor of Mathematics in University College, London, and began lecturing at the early age of 22 years.

But the regents of this College claimed the right of dismissing a professor without assigning reasons, and acted upon this principle in dismissing the Professor of Anatomy. Immediately De Morgan resigned in protest. After the regulations were changed he was invited to resume his chair, was reappointed, and served for the next 30 years. His dislike to honorary titles led him to refuse the offer of LL. D. from the University of Edinburgh.

In 1866 a discussion arose as to the true interpretation of the principle of religious neutrality avowedly adopted by the College. De Morgan held that any consideration of a candidate's ecclesiastical position or creed or lack of creed was inconsistent with the principle. In protest against a violation of this principle he resigned a second time in a letter dated November 10th, 1866. He was always remarkably free from any touch of sordid self-interest.

As a teacher, De Morgan was particularly gifted. A voluminous writer on mathematics, he contributed essentially to those expansions of the fundamental concepts which have rendered possible the new algebras, such as Quaternions, and have generalized the whole idea of a mathematical algorithm, until now an American produces a thesis entitled "Mathematics the Science of Algorithms."

His logical work alone would give De Morgan lasting fame. Here he stands alongside his immortal contemporary, Boole. The eternally memorable year in the history of Logic was 1847, in which George Boole issued "The Mathematical Analysis of Logic, being an Essay toward a Calculus of Deductive Reasoning," von Staudt published his "Geometrie der Lage," a mathematics utterly freed from any idea of quantity, a mathematics strictly qualitative, and De Morgan printed his fundamental treatise, called "Formal Logic.; or, the Calculus of Inference, Necessary and Probable."

The great memoirs produced in 1850, 1858, 1860, 1863 are preserved, if buried, in the inaccessible "Cambridge Philosophical Transactions." De Morgan's great combination of logical with mathematical learning, and his prominent position in London, the great metropolis, made him the man to whom resorted all the Circle-Squarers, Angle-Trisectors, Perpetual-Motionists, Triangle-Anglesumers. Adding this curious experience to his great bibliographic knowledge of what had been attempted in that way in the past, he formed a large book called "A Budget of Paradoxes," which is one of the most interesting treatises ever written on what may be called extended fallacies. This charming book has steadily advanced in market price until I find it cited in Macmillan's Catalogue No. 245 (1086) at fifty shillings per copy.

It was De Morgan who first gave a thorough treatment of contrary, nega-

tive, or contradictory terms, though in just the sense I have heard babies use them while learning our language. But remember it is Darwinism that has since taught the world to learn at the feet of babes. Bain says: "According to the true view of contrariety, as given by De Morgan, the negative is a remainder, gained by the subtraction of the positive from the universe; the negative of X is $U-X$, and may be symbolized by a distinct mark, x; whence X and x are the opposites under a given universe; not $-X$ is x, and not$-x$ is X."

Of the separation of logic and mathematic De Morgan says: "The effect has been unfortunate. . . . The sciences of which we speak may be considered either as disciplines of the mind, or as instruments in the investigation of nature and the advancement of the arts. In the former point of view their object is to strengthen the power of logical deduction by frequent examples; to give a view of the difference between reasoning on probable premises and on certain ones by the construction of a body of results which in no case involve any of the uncertainty arising from the previous introduction of what may be false; to establish confidence in abstract reasoning by the exhibition of processes whose results may be verified in many ways; to help in enabling to acquire correct notions of generalization; to give caution in receiving that which at first sight appears good reasoning; to instill a correct estimate of the powers of the mind by pointing out the enormous extent of the consequences which may be developed out of a few of its most fundamental notions; and to give the luxury of pursuing a study in which self-interest cannot lay down premises nor deduce conclusions.

As instruments in the investigation of nature and the advancement of the arts it is the object of these two sciences to find out truth in every matter in which nature is to be investigated, or her powers and those of the mind to be applied to the physical progress of the human race, or their advancement in the knowledge of the material creation."

De Morgan was fond of laughing at the metaphysicians. He says: "We know all about *can* and *cannot* from our cradles; we never feel the same assurance about *is* and *is not*.

A philosopher, in a dark age, may determine to set out with a knowledge of the naturally possible and impossible; but not even a philosopher ever pretended to set out with a knowledge of the existent and non-existent."

Aristotle and all the old logicians said that the whole of the middle term must be taken in at least one of the premises. As they put it, the middle term must be distributed at least once in the premises, otherwise the minor term may be compared with one part and the major with another part of it. From

Some men are poets,
Some men are Indians,

nothing follows. But the Aristotelians were wrong, as De Morgan clearly showed in his doctrine of Plurative Judgments. For example, if we have given the premises,

Most men are uneducated,
Most men are superstitious,

according to Aristotle we are not warranted in drawing any conclusion; for the middle term is men, and in neither premise is anything said about all men.

But, in point of fact, we can draw the perfectly valid conclusion,

Some uneducated men are superstitious.

Again Aristotle is contradicted by numerically definite judgments. In these there is inference when the quantities of the middle term *in the two premises together* exceed the whole quantity of that term.

Lambert first thought of this principle. De Morgan reconceived it and extended its use.

Suppose we grant the premises,

Two-thirds of all adults are women.

The number of women who have been married is never greater than the *total* number of men. It follows that half the entire number of women are single.

Still, easy and certain as such reasoning is, it may be difficult to a logician trained only in the traditional logic.

In a Princeton "Manual of Logic" the only numerically definite syllogism given was erroneous, and stood so for years. I stated this to the author, and in the latest stereotyped edition it has been changed. The Syllogism he gave was as follows:

"60 out of every 100 are unreflecting.
"60 out of every 100 are restless.
"Therefore, 20 out of every 100 restless persons are unreflecting."

After pointing out to him the fault in what he had been teaching for years, the following has been substituted:

"60 out of this 100 are unreflecting.
"60 out of this 100 are restless.
" $\therefore$ 20 restless persons are unreflecting."

But De Morgan's greatest work was connected with his development of the Logic of Relatives, independently discovered by Robert Leslie Ellis after reading Boole's "Laws of Thought."

One of De Morgan's last memoirs, in the tenth volume of the "Cambridge Transactions," was on the Logic of Relations, which is, in the mathematical sense, a far-reaching generalization of the old logic. In our modern mathematics everything is generalized as far as possible. Every study of a generalization gives additional power over the particular. We need to go beyond and look back from an elevation.

Any first-rate mathematician working in logic would attempt to generalize, and, in fact, Boole generalized the scholastic logic in a manner entirely different from De Morgan. In De Morgan's view of the subject, the purely formal proposition with judgment wholly void of matter, is seen in "There is the probability x that X is in the relation L to Y." The syllogism is the determination of the relation which exists between two objects of thought by means of the relation

in which each of them stands to some third object which is the middle term. The pure general form of the syllogism, when its premises are absolutely asserted, is as follows: X is in the relation L to Y. Y is in the relation M to Z; therefore X is in the relation "L of M," compounded of L and M, to Z. In ordinary logic the actual composition of the relation is made by our consciousness of its *transitive* character. A relation is transitive when, being compounded with itself, it reproduces itself; that is, L is transitive when every L of L is L; for example "brother."

Thus De Morgan broke away from that paltry narrowness which asserts that our minds in pure thinking can use nothing but the relation of identity; from the Jevons sophism that thought cannot move because all thought is the substitution of identicals.

So we see that in logic, as in mathematics, we may develop a whole system of theorems about symbols which are to be used in a given manner; and then to make this whole system true of a desired relation or subject matter we have only to show that this relation or subject matter fulfills the few fundamental principles of the system.

De Morgan treated of convertible and inconvertible relatives, repeating relatives; non-repeating relatives, transitive and intransitive relatives, and inaugurated a general system.

To what tremendous estate this system has grown may be seen in the 649 pages of Ernst Schroeder's Treatise, "Algebra und Logik der Relative;" Leipzig, Teubner, 1895, on whose first page, as founder of the system, stands the name of Augustus De Morgan.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

One of the most important applications of substitution groups occurs in the theory of the solution of algebraic equations. It seems desirable that a fairly complete discussion of the solution of the quadratic equation should precede the study of this application. It is hoped that this discussion will not be without interest in itself even if the facts with which we have to deal are well known. As we shall need a clear idea of the *domain of rationality* we shall first develop several elementary concepts which involve the notion of groups and naturally lead to the more general concept of the domain of rationality.

Let us first consider the totality of numbers (T_1) formed by all the positive integers,* each positive integer occurring once and only once. By adding

*We shall throughout confine our attention to the finite. Not only are the numbers to be regarded as finite but the number of times that a given operation is to be performed is also to be considered finite.

any one of these to itself or to any other number in T_1 we obtain no new number. We may therefore say that T_1 forms a *group* (G_1) *with respect to addition*. It is evident that T_1 also forms a group (G_2) with respect to multiplication. Each of these two groups contains an indefinite number of subgroups.* To every subgroup of G_1 there corresponds a subgroup of G_2 which contains the same numbers but the converse is not true. For instance, all the positive integral powers of any prime number form a subgroup of G_2† but they do not form a subgroup of G_1.

We shall not enter upon the discussion of the subgroups found in G_1 and G_2 but only call attention to a few of the most simple ones. All the even positive integers clearly form a subgroup of both of these groups. We may inquire what is the smallest subgroup that contains any given positive integer a. In G_2 this subgroup is a^α, $\alpha=1, 2, 3, \ldots\ldots$ while in G_1 this subgroup is βa, $\beta=1, 2, 3, \ldots\ldots$ Hence unity is a subgroup of G_2 but not of G_1. As G_1 contains no subgroup that involves unity we may say that it is generated by this element. G_2 is generated by all the prime numbers together with unity.

The totality of negative integers (T_2 also forms a group with respect to addition but it does not form a group with respect to multiplication. It is clear that the smallest group with respect to multiplication that contains T_2 must also contain T_1. That is, $T_1+T_2=T_3$ is a totality which forms a group with respect to multiplication. T_3 also forms a group with respect to each of the operations addition and subtraction. The group with respect to subtraction does not contain any subgroup involving either T_1 or T_2, that with respect to multiplication contains a subgroup that involves T_1 but none that involves T_2, while that with respect to addition contains a subgroup that includes T_1 and also one that includes T_2.

Among the numbers which are now in common use those included in T_1 are perhaps of special importance as is also indicated by the fact that they are frequently called the natural numbers. In considering the groups which certain totalities of numbers form with respect to given operations it is therefore of special importance to inquire into the smallest groups that contain T_1. We have already seen that with respect to subtraction this smallest group includes other numbers than those contained in T_1. Similarly we observe that with respect to division‡ this smallest group includes an additional totality of numbers, viz: the fractions whose numerators and denominators are positive integers.

Instead of inquiring into the smallest totality of numbers that contains T_1 and forms a group with respect to a given operation we may also inquire into the smallest totality that contains T_1 and forms a group with respect to each of a given number of different operations. For instance, the smallest totality

*The term subgroup is used, as usual, to represent a group that is contained in the larger group under consideration.

†For brevity we shall say that certain elements of a group form a subgroup instead of saying that these elements form a group if they are combined according to the same operation as the elements of the group.

‡As we have excluded the infinite we are not allowed to divide by 0. We may regard this as an impossible operation in the region to which we confine our observations.

(T_4) that contains T_1 and forms a group with respect to each of the four fundamental operations—addition, subtraction, multiplication, and division—is that which is formed by all the rational numbers. If we form the smallest totality that contains unity and forms a group with respect to each of these four operations we evidently arrive again at T_4. On this account the totality of all the rational numbers is generally called the domain unity. This is the simplest domain of rationality.*

It would be of interest to consider all the different types of subgroups of the groups formed by T_4 with respect to the given fundamental operations. We shall not enter into this field as the matter has probably been sufficiently developed for our present purposes. We would remark, however, that a careful study of these matters seems to us to be one of the simplest roads towards forming a clear notion of groups as well as of the domain of rationality.

The operations which we have thus far considered may be represented by the simple equations:

$$a+b=x, \quad a-b=x, \quad a\times b=x, \quad a\div b=x.$$

We have seen that x is always a rational number when both a and b are such numbers. In other words, we have seen that when we take a and b from the totality of numbers represented by T_4, x will also belong to this totality, but when we take a and b from one of the other totalities of numbers that have been considered—T_1, T_2, T_3—we cannot affirm that x belongs to the same totality in all the equations.

At an early stage in the development of mathematics it became necessary to solve equations of a higher than the first degree. One of the great difficulties which presented itself at this point was the fact that T_4 does not form a group with respect to the algebraic operations whose degree exceeds one. As long as these operations can be represented by a binomial equation of the form

$$x^n=a$$

n being a positive integer and a being a positive rational number, the difficulty was not much greater than that which had to be overcome at preceding stages, for the extension of the totality of numbers in such a way as to include irrational numbers does not seem an irrational adventure, even if the association of these numbers with matters of observation is not so direct and evident as it had been in the preceding cases.

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*The totality of rational functions with integral coefficients of a given number of determinate or indeterminate independent quantities (R, R', R'', ...) is called, after Kronecker, a domain of rationality. In other words, it is the smallest totality that contains these quantities and forms a group with respect to the four fundamental operations. In the domain unity we have clearly only one such quantity and this is determinate, viz: $R=1$. We shall soon consider a domain of one indeterminate quantity or parameter. Sometimes the given R's are defined as indeterminate parameters. According to this definition the domain unity contains no parameter. This domain is clearly generated by any rational finite number except 0. It may therefore be called the domain 2, 3, ... as well as the domain 1.

A much more serious difficulty presented itself when it was required to introduce the operation indicated by the general quadratic equations

$$x^2+bx+c=0. \qquad (A).$$

Even if we take b and c from the totality T_1 it often happens that we can not find any number among those that have been considered which comes near towards satisfying this equation. Hence the totality of real numbers (T_5) can clearly not form a group with respect to this operation. With respect to the four fundamental operations T_5 forms a group which contains T_4 as a subgroup.

It should however not be inferred that none of the numbers that have been considered form a group with respect to the operation (A). In order that a single number (α) may have this property it is necessary and sufficient that $x=\alpha$ satisfies the equation

$$x^2+\alpha x+\alpha=0.$$

Hence there are two numbers ($-\frac{1}{2}$ and 0) each of which forms a group with respect to the quadratic equation. 0 also forms a group with respect to the four fundamental operations but $-\frac{1}{2}$ does not have this property. It should also be observed that when $\alpha=-\frac{1}{2}$ it is not the only value of x that satisfies the given equation and that the term group has therefore to used in a somewhat restricted sense when we say that $-\frac{1}{2}$ forms a group with respect to the quadratic equation.

But, even if it was known that certain special numbers form a group with respect to the operation (A), and, what is more important, that x belongs to the totality of numbers T_6 for a large number of types of (A) yet the matter remained in an unsatisfactory state as long as no totality of numbers was known which includes T_5 and forms a group with respect to (A). The struggle for light on this point was a long one, reaching far into our century. We cannot enter into a history of this struggle. It must suffice to state that the triumph was largely due to the elegant geometrical representation of the complex number by means of points in the plane. This was not the first nor last assistance that algebra has received from his sister geometry. On the other hand, algebra has a very brilliant record of services rendered to his ambitious sister.

The importance of the adoption of the complex numbers (T) cannot be fully appreciated if we confine our attention to the quadratic equation. If the general algebraic operations of each of the following degrees had again required equally great extension in the number system this would soon have become exceedingly difficult and the progress in the solution of the algebraic equations could not have been so rapid. The great importance of the adoption of the totality of numbers T may therefore be said to be due to the fact that it forms a group with respect to the general algebraic operation indicated by the following equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_n=0 \qquad (B),$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ belong to T and n belongs to T_1. In other words if the coefficients of (B) belong to the domain of the indeterminate parameter

$$R=\sqrt{a} \qquad (a \text{ belonging to } T_5)$$

x will also belong to this domain. Or again, B can be resolved into factors without using numbers that lie outside of T.

While we cannot fully appreciate the importance of the adoption of the complex numbers when we confine our attention to the quadratic equation, yet in this equation we see the source of T and with it we naturally associate the wonderful progress achieved by means of T. It is however not our object to enter upon the discussion of the important position which the quadratic equation occupies in the development of mathematics even if an idea of this position naturally increases the interest in this equation and hence also in its discussion as an algebraic operation.

To convey an idea of the importance of the concept domain of rationality we shall consider an application. Suppose that we have an equation of the form (B) and that its coefficients belong to a certain domain of rationality T' while none of its factors belong to this domain.* Suppose that we have any other equation whose coefficients also belong to T' and that these two equations have a common root. We can then say that the first equation is a factor of the second. In other words, we know that all the roots of the first equation are also roots of the second. The truth of this statement follows directly from the fact that the coefficients of the greatest common divisor of the two equations must belong to T'. This greatest common divisor must therefore be the first member of the first equation.

If we have any complex number

$$a+bi$$

the quadratic equation which contains this number and its conjugate for its roots is

$$x^2-2ax+a^2+b^2=0. \qquad (C).$$

The coefficients of this equation belong to T_5. Suppose now that we have any other equation that contains $a+bi$. From the proof just given it follows that it must also contain (C). The same remarks clearly apply to surd roots of the form

$$a\pm\sqrt{b}.$$

Hence we see that the given statement includes the theorems that if an equation with real coefficients contains the root $a+bi$ it also contains the root $a-bi$, and if an equation whose coefficients are rational contains a surd root of the form $a+\sqrt{b}$ it also contains $a-\sqrt{b}$ as a root.

*In this case we say that the equation is irreducible in the domain T'.

THE NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from May Number.]

SCHOLION.

Nevertheless it might be doubted, whether, from whatever point K (assumed indeed in BX before the meeting of this BX with the other AX) erected toward the parts of the straight AX, a perpendicular must meet this (fig. 29) in some point L; provided of course those two, before the aforesaid meeting, are assumed ever more to approach each other mutually [and not to meet at any finite remove].

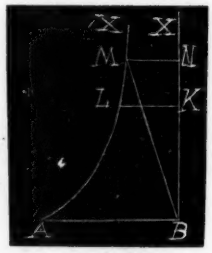


Fig. 29.

But I say it will follow completely thus.

Proof. Let there be assigned in BX any point whatever K. In AX is taken a certain AM equal to the sum of this BK and of twice AB.

Then from the point M is drawn to BX (according to Eu. I. 12) the perpendicular MN. According to the present supposition, MN will be less than AB. Wherefore AM (made equal to the sum of BK and of double AB) will be greater than the sum of BK, AB, and NM. Now it behooves to show this same AM to be less than the sum of BN, AB, and MN, that thence it may follow this BN is greater than the aforesaid BK, and therefore the point K lies between the points B and N.

Join BM. The side AM will be (from Eu. I. 20) less than the two remaining sides together AB and BM. Again the side BM (from the same Eu. I. 20) will be less than the two sides together BN and MN. Therefore the side AM will be by much less than the three sides together AB, BN, and NM. But this was to be shown, in order to deduce that the point K lies between the points B and N. Thence however it follows, that the perpendicular from the point K erected toward the parts of AX must meet this in some point L stationed between the points A and M; else obviously (against Eu. I. 17) it must cut either AB or MN perpendiculars to BX.

Quod etc.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from December Number.]

XXII. Let ABC be a triangle, right-angled at C. Assume $CB<CA$. Complete the figure as indicated, the point E being at the middle of AB.

There are three cases: (1) when E is without the circle, (2) in the circumference, (3) within the circle. We treat but one case, when E is within the circle. The other cases may be treated in a similar manner.

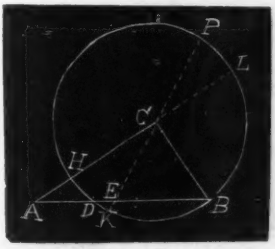


Fig. 17.

$$AD=\frac{AH\cdot AL}{AB}=\frac{b^2-a^2}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$DE=\frac{KE\cdot EP}{EB}=\frac{(a-\frac{1}{2}c)(a+\frac{1}{2}c)}{\frac{1}{2}c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Add (1) and (2), $\frac{1}{2}c=\frac{a^2+b^2-\frac{1}{2}c^2}{c}$. $\therefore\ c^2=a^2+b^2$.

NOTE. This method is credited, by Wipper, to Krueger, 1746. Though in the original, the legs are arbitrarily assumed as unequal, we may pass, by the theory of limits, to the case when the legs are equal.

XXIII. Let ABC be a triangle, right-angled at C. Circumscribe the triangle by a circle. Complete the rectangle. Then,

$$AB\cdot CD=CB\cdot AD+AC\cdot DB, \text{ or } c^2=a^2+b^2.$$

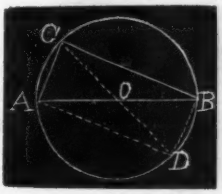


Fig. 18.

NOTE. Thought this method must have been known independently to many persons, it seems that its first appearance in print was in 1859, in Runkle's *Mathematical Monthly*.

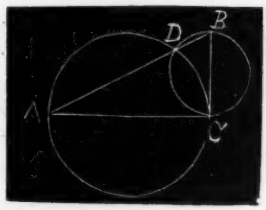


Fig. 19.

XXIV. Let ABC be a triangle, right-angled at C. Construct circles with AC and BC as diameters, respectively. These circles will intersect in AB, as at D.

Then, $\overline{AC}^2=AD\cdot AB$, and $\overline{BC}^2=BD\cdot AB$.

Add, $\overline{AC}^2+\overline{BC}^2=AB(AD+BD)=\overline{AB}^2$.

NOTE. This is one of Richardson's.

XXV. Let ABC be a triangle, right-angled at C. Complete the figure as indicated.

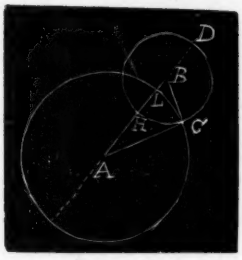


Fig. 20.

Then, $\overline{AC}^2 = AH \cdot AD$, and $\overline{BC}^2 = BL \cdot BE$.

Add, $\overline{AC}^2 + \overline{BC}^2 = AH \cdot AD + BL \cdot BE$

$= (AB - BC)(AB + BC) + (AB - AC)(AB + AC)$.

$\therefore \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Richardson's method is somewhat different. Thus:

$$\overline{AC}^2 + \overline{BC}^2 = AH(AB + BC) + BL(AB + AC)$$

$$= AB(AH + BL) + AH \cdot BC + BL \cdot AC + BL \cdot AB - BL \cdot AB$$

$$= AB(AH + BL) + AH \cdot BC + BL(BL + 2AC) - BL \cdot AB$$

$$= AB(AH + BL) + AH \cdot BH + \overline{BH}^2 - BL(AH + BH)$$

$$= AB(AH + BL) + (AH + BH)(BH - BL)$$

$$= AB(AH + BL) + AB \cdot HL = AB(AH + BL + HL) = \overline{AB}^2.$$

XXVI. Fig. 20.

$EH : ED :: HL : DL$. (See Olney, §971). $\therefore EH \cdot DL = ED \cdot HL$,

or $(AC + AB - BC)(BC + AB - AC) = (AC + AB + BC)(AC + BC - AB)$.

$\therefore \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield; Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

70. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years; when can I pay him \$100 to settle the account equitably, money being worth 6%?

I. Solution by the PROPOSER, and P. S. BERG, Larimore, North Dakota.

Let x=the time.

Now the amount of \$200 for $(x-4)$ years—the amount of \$100 for $(x-2)$ years must=\$100.

$200+12(x-4)=152+12x$=amount of \$200 for $(x-4)$ years at 6%.

$100+6(x-2)=88+6x$=amount of \$100 for $(x-2)$ years at 6%.

$\therefore (152+12x)-(88+6x)=100$. $\therefore x=6$ years.

II. Solution by Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Computing at simple interest,

$\$\frac{200}{134}-\$\frac{100}{112}=\$72\frac{1}{17}$, amount equitably due now.

Hence, $\$100-\$72\frac{1}{17}=\$27\frac{16}{17}$ to be earned as interest.

$\$27\frac{16}{17}\div(72\frac{1}{17}\times\frac{6}{100})=5\frac{31}{75}$, the number of years required.

III. Solution by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

In this I assume interest to remain unpaid until the time of settlement, and to *draw no interest*.

A owes me to-day the present worth of \$100 due in 2 years at 6%, or \$89.29.

I owe A to-day the present worth of \$200 due in four years at 6%, or \$161.29.

That is, I owe A \$72 more than he owes me. Hence the problem reduces itself to the question, when will the excess of my interest over his plus \$72 amount to \$100? That is, my interest must exceed his by \$28.

My yearly excess is \$4.32. Hence to gain \$28, 6.481 years will be required.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

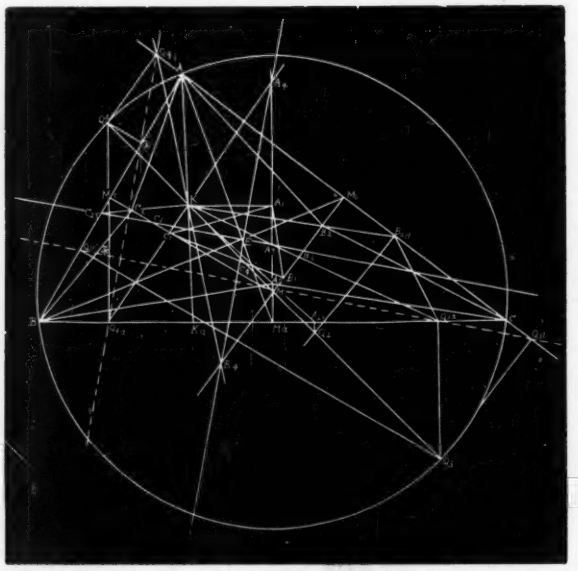
SOLUTIONS OF PROBLEMS.

65. Proposed by I. J. SCHWATT, Ph. D., Professsor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

The axes of the ellipse isogonal to Lemoine's line with respect to a triangle (Steiner's ellipse), are parallel to Simson's lines belonging to the extremities of Brocard's Diameter.

Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Let the triangle be ABC; center of circumcircle, M; $A_1B_1C_1$, the Brocard triangle with vertices on parallels thro' Grebe's point and perpendiculars at mid-points of sides of ABC. It is known that ABC and $A_1B_1C_1$ are similar. Let E be the medio-centre, or centre of gravity, of ABC. It is known that E is also medio-centre of $A_1B_1C_1$. Let MK, Brocard's diameter (or the diameter of circle about Brocard's triangle) be produced to meet circumcircle of ABC in the points Q_3 and Q_4.



By the construction, the vertices of Brocard's triangle are also the vertices of three similar isosceles triangles; for these isosceles triangles have as altitudes the perpendiculars from Grebe's point upon the sides of ABC, and it is known that these perpendiculars are as the sides. Hence the triangles have bases and altitudes proportional, and therefore are similar.

If, now, any three similar isosceles triangles be constructed upon the sides of ABC, their vertices A_2, B_2, C_2, will be the vertices of a triangle having the same medio-centre as ABC or $A_1B_1C_1$. The proof of this is similar to that which is known to establish E the same for ABC, and $A_1B_1C_1$.

Draw KA_2, KB_2, KC_2 to meet sides of triangle ABC in points $A_{2\alpha}$, $B_{2\beta}$, $C_{2\gamma}$ respectively. Then triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ is similar to the triangle $A_2B_2C_2$, and centre of similitude is K. This may be proved as follows: Erect a perpendicular at $A_{2\alpha}$ to cut Brocard's diameter (produced) at Q_2, then

$$A_2M_a : KK_a = A_{2\alpha}A_2 : A_{2\alpha}K = Q_2M : Q_2K. \qquad (1).$$

Triangles A_2BC, and B_2AC are similar by construction, hence

$$A_2M_a : B_2M_b = M_aC : M_bC = a : b = A_1M_a : B_1M_b,$$

where $a : b$ is ratio of two sides of triangle ABC.

We may write this last

$$A_2M_a : A_1M_a = B_2M_b : B_1M_b = B_{2\beta}B_2 : B_{2\beta}K = Q_2M : Q_2K \qquad (2).$$

From (1) and (2) we have,

$$A_{2\alpha}A_2 : A_{2\alpha}K = B_{2\beta}B_2 : B_{2\beta}K,$$

hence the lines A_2B_2 and $A_{2\alpha}B_{2\beta}$ become parallel, and if the same course of reasoning be pursued with regard to the other sides, the triangles are seen to be similar, with K the center of similitude.

Now, from the equality, $B_{2\beta}B_2 : B_{2\beta}K = Q_2M : Q_2K$, it follows that $B_{2\beta}Q_2$ is parallel to B_2M; and since B_2M is perpendicular to AC, $B_{2\beta}Q_2$ is also perpendicular to AC. Similarly, the perpendicular to AB at $C_{2\gamma}$ passes through Q_2, and we have already that the perpendicular to BC at $A_{2\alpha}$ passes through Q_2. If Q_2 be caused to coincide with either Q_3 or Q_4, then triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ will degenerate into the straight lines $Q_{3a}Q_{3b}Q_{3c}$, and $Q_{4a}Q_{4b}Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4. Also triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$, and $A_4B_4C_4$, which are parallel to Simson's lines, to Q_3, Q_4. These lines will also pass through the medio-centre, E, since the triangles which degenerate continually have E as the medio-centre. Since the Simson lines are perpendicular to each other (see Geometry of Simson lines), these last mentioned lines through E, are perpendicular to each other. Since we know that the ellipse (Steiner's) has these lines for axes, the proposition is proved. Q. E. D.

NOTE. The above solution I got from Dr. Schwatt. An elegant demonstration of properties of the ellipse is given in Schwatt's Isogonal Curves, (Leach, Shewell & Sanborn, New York). F. M. M.

This problem was also solved by Prof. G. B. M. Zerr. Prof. William Hoover did not solve it, but referred to the proof given in *Casey's Analytical Geometry*, Edition of 1893, Articles 364, 365 (Cor. 1).

66. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is an equilateral hyperbola.

I. Solution by the PROPOSER.

By Salmon's *Conic Sections*, Sixth Edition, Ex. 4, page 234, the equation to the circle osculating a parabola $y^2 = px$ .....(1) at (x', y') is

$$(p^2+4px')(y^2-px) = \{2yy'-p(x+x')\}\{2yy'+px-3px'\} \ldots\ldots \ldots(2).$$

At the vertex, $x'=0$, $y'=0$, and (2) becomes

$$x^2+y^2-px=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If (x_1, y_1) be any point on the required locus, its polar with respect to (1) is

$$px-2y_1y+px_1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The condition that (4) touches (3) is

$$x_1^2-y_1^2-px_1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

an equilateral hyperbola.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; CHAS. C. PURYEAR, Professor of Mathematics, Agricultural and Mechanical College, College Station, Texas; and G. B. M. ZERR, Texarkana, Ark.

Let $y^2=4ax$ be the equation to the parabola, (b, c) any point.

Then $cy=2a(x+b)\ldots\ldots(1)$ is the polar of (b, c). $x^2+y^2=ax\ldots\ldots(2)$ is the circle of curvature at the vertex. The value of y from (1) in (2) gives

$$c^2x^2+4a^2x^2+8a^2bx+4a^2b^2=ac^2x\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (3) we find the condition that (1) should be tangent to (2) to be

$$a^2c^4=8a^3bc^2+16a^2b^2c^2. \quad \therefore c^2=8ab+16b^2.$$

$$\therefore a^2=(4b+a)^2-c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

(4) represents an equilateral hyperbola.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

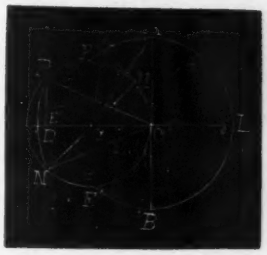
55. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of grouhd b feet in diameter. If the horse is tethered outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? b is greater than a in each case.

I. Solution by the PROPOSER.

Consider area $C'HFMA$. $C'A=C'P=a$, feet. $CC'=CH=\frac{1}{2}b$, feet. Let $\angle HCF=\phi$. $HM=$arc $HF=\frac{1}{2}b\phi$. The area element is $\frac{1}{2}(HM)^2 d\phi$.

$$\therefore \text{ area } C'HFMA=\tfrac{1}{8}\int_0^{\frac{2a}{b}} b^2\phi^2 d\phi,=\frac{a^3}{3b}.$$



Hence, when the horse is outside, he can graze over

$$\frac{a^2}{6b}(4a+3b\pi), \text{ square feet.}$$

Draw PE at right angles to DC'. Let $x=CE$, and $\angle PC'E=\theta$.

$$\text{Area of sector } PC'D=\int_0^{\theta}\int_0^{a} dad\theta,=\tfrac{1}{2}a^2\cos^{-1}\frac{b+2x}{2a},=\text{sector } NC'D.$$

$$\text{Area of sector } PCC'HF=\tfrac{1}{8}b^2\times\text{angle } PCC',=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right).$$

$$\text{Area of triangle } PCC'=\tfrac{1}{8}b\sqrt{b^2-4x^2}.$$

$$\therefore \text{ area of segment } PFHC'=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right)-\tfrac{1}{8}b\sqrt{b^2-4x^2},=\text{segment } HC'F.$$

$$x=\frac{2a^2-b^2}{2b}.$$

Hence, area of $C'HPNF'=a^2\cos^{-1}\frac{a}{b}+\frac{1}{4}b^2\cos^{-1}\frac{b^2-2a^2}{b^2}-\frac{1}{2}a\sqrt{b^2-a^2}$, square feet ; the space grazed over inside the fence.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

In first part required area equals that of semicircle $ALB+2\times AMFHC$. Let HM be a portion of rope unwound and equal to HF. Let $\angle HCF=\phi$, $\angle MCF=\theta$, $CM=\rho$, and take CF for polar axis. Then

$$\rho^2=\tfrac{1}{4}b^2+\overline{HM}^2,=\tfrac{1}{4}b^2+\tfrac{1}{4}(b^2\phi^2).$$

$$\phi=\frac{2}{b}\sqrt{4\rho^2-b^2}. \qquad \frac{b}{2\rho}=\cos(\phi-\theta),=\cos\left(\frac{2}{b}\sqrt{4\rho^2-b^2}-\theta\right).$$

$$\theta=\frac{2}{b}\sqrt{4\rho^2-b^2}-\cos^{-1}\left(\frac{b}{2\rho}\right).$$

$$\frac{d\theta}{d\rho} = \frac{4\rho}{b\sqrt{4\rho^2-b^2}} - \frac{b}{2\rho^2\sqrt{1-\frac{b^2}{4\rho^2}}}, \quad = \frac{\sqrt{4\rho^2-b^2}}{b\rho}.$$

$$\frac{dA}{d\rho} = \frac{dA}{d\theta}\frac{d\theta}{d\rho}, \quad = \frac{\rho^2}{2} \times \frac{\sqrt{4\rho^2-b^2}}{b\rho}, \quad = \frac{\rho\sqrt{4\rho^2-b^2}}{2b}.$$

Limits of ρ for $CFMA$ are seen to be $\frac{1}{2}b$ and $\sqrt{a^2+\frac{1}{4}b^2}$.

$$\therefore \text{ Area } CFMA = \frac{1}{2b}\int_{\frac{1}{2}b}^{\sqrt{a^2+\frac{1}{4}b^2}} \rho\sqrt{4\rho^2-b^2}\, d\rho, \quad = \frac{a^3}{3b}.$$

Area $FHCAM = CFMA + C'CA - CFHC', = a^3/3b + ab/4 - ab/4 = a^3/3b$.

$\therefore$ Area $F'BLAFHC' = 2a^3/3b + \pi a^2/2$.

Internal area is composed of $2\times$ segment $PHC' +$ sector $PDNC'$

$$\sin\tfrac{1}{2}\angle PCC' = (\tfrac{1}{2}a/\tfrac{1}{2}b) = a/b. \quad \angle PCC' = 2\sin^{-1}(a/b).$$

$$\angle PCN = 2\pi - 4\sin^{-1}(a/b). \quad \angle PC'N = \pi - 2\sin^{-1}(a/b).$$

$$\text{Sec. } PDNC' = \frac{a^2}{2}\left(\pi - 2\sin^{-1}\frac{a}{b}\right), \text{ sector } CPHC' = \frac{b^2}{8}\left(2\sin^{-1}\frac{a}{b}\right), = \frac{b^2}{4}\sin^{-1}\left(\frac{a}{b}\right)$$

$$\triangle PCC' = \tfrac{1}{2}a\sqrt{\tfrac{1}{4}b^2 - \tfrac{1}{4}a^2}, = \tfrac{1}{4}a\sqrt{b^2-a^2}.$$

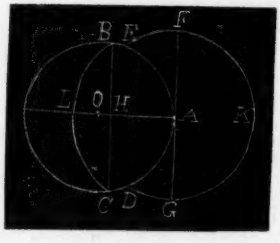
$$\text{Segment } PHC' = \frac{b^2}{4}\sin^{-1}\frac{a}{b} - \tfrac{1}{4}a\sqrt{b^2-a^2}.$$

$$\therefore \text{ Internal area } = \frac{a^2}{2}\left(\pi - 2\sin^{-1}\frac{a}{b}\right) + \frac{b^2}{2}\sin^{-1}\frac{a}{b} - \tfrac{1}{2}a\sqrt{b^2-a^2},$$

$$= \tfrac{1}{2}\pi a^2 - \frac{a}{2}\sqrt{b^2-a^2} + (\tfrac{1}{2}b^2 - a^2)\sin^{-1}\left(\frac{a}{b}\right).$$

III. Solution by G. B. M. YERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A be the point where the horse is tethered. $AF = a$, $AO = b/2$. Area $EADGKFE = 2$ area $EAF +$ area of semicircle GKF.



$$\therefore A = \int \rho^2 d\theta + \tfrac{1}{2}\pi a^2 \text{ ; but } \rho = \tfrac{1}{2}b\theta.$$

$$\therefore A = \tfrac{1}{4}b^2\int_0^{2a/b} \theta^2 d\theta + \tfrac{1}{2}\pi a^2, = \frac{a^2}{6b}(4a + 3\pi b).$$

Let $x^2 + y^2 = \frac{1}{4}b^2$, be the equation to circle center O. $(x - \frac{1}{2}b)^2 + y^2 = a^2$, be the equation to circle center A.

$$\therefore OH=\frac{b^2-2a^2}{2b},\quad \therefore BH=\frac{a}{b}\sqrt{b^2-a^2}.$$

A'=area of segment BLC+area of segment BAC,

$$=\frac{b^2}{4}\left\{\sin^{-1}\left(\frac{2a}{b^2}\sqrt{b^2-a^2}\right)-\frac{2a(b^2-2a^2)\sqrt{b^2-a^2}}{b^4}\right\}$$

$$+a^2\left\{\sin^{-1}\frac{\sqrt{b^2-a^2}}{b}-\frac{a\sqrt{b^2-a^2}}{b^2}\right\},$$

$$=\tfrac{1}{4}b^2\cos^{-1}\left(\frac{b^2-2a^2}{b^2}\right)+a^2\cos^{-1}\frac{a}{b}-\frac{a}{2}\sqrt{b^2-a^2}.$$

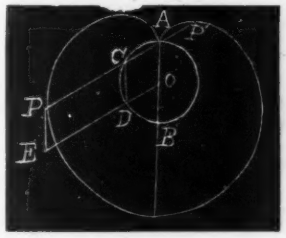
Also solved by *J. SCHEFFER* and *A. H. HOLMES*.

56. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the center of gravity of the solid from the origin O; and (7) the distance g_0 of the center of gravity of the plane curve from the origin O.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let $AB=a$ be the diameter of a circle. From A draw any chord AC. Make CP and $CP'=b$, then will the locus of P or P' be the Limaçon. If $AP=r$, $\angle PAB=\theta$, we find at once the polar equation of the Limaçon to be $r=a\cos\theta+b$. If $b>a$, the curve consists of but one loop; if $b<a$, it has two loops, and if $b=a$ the curve becomes the Cardioid, the polar equation of which is $r=a(1+\cos\theta)$. It can easily be shown that the cardioid is an epicycloid, the generating circle of which is equal to the fixed one; also, drawing through the center O of the circle a line parallel to AP cutting the circumference of the circle at D, and drawing through P a line parallel to CD, this line is a tangent to the cardioid at P. The different problems proposed are best solved by means of the polar equation of the curve.



(1). The length $s=2\int_0^\pi d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$, $=2a\int_0^\pi\cos\tfrac{1}{2}\theta d\theta$, $=8a$.

(2). The area $A=\int_0^\pi r^2d\theta$, $=a^2\int_0^\pi(1+\cos\theta)^2d\theta$, $=\frac{3}{2}\pi a^2$.

(3). To find the maximum ordinate, $r\sin\theta = a(\sin\theta + \frac{1}{2}\sin 2\theta$ is to be a maximum. By differentiation we find $\theta = 60°$. $\therefore$ maximum ordinate $= \frac{3}{4}a\sqrt{3}$, and circumference $C = \frac{3}{2}\pi a\sqrt{3}$.

(4). Surface $K = 2\pi\int_0^\pi r\sin\theta d\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)}$,

$$= 4\pi a^2\int_0^\pi (1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta d\theta, = -16\pi a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta, = \frac{32\pi a^2}{5}.$$

For the distance x_0 of the center of gravity of this surface we have

$$Kx_0 = 2\pi\int_0^\pi r^2\sin\theta\cos\theta(ds/d\theta).d\theta, = 4\pi a^3\int_0^\pi (1+\cos\theta)^2\sin\theta\cos\theta.\cos\tfrac{1}{2}\theta d\theta,$$

$$= -64\pi a^3\left[2\int_0^\pi \cos^8\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta.\int_0^\pi \cos^6\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta\right], = \frac{320\pi a^3}{63};$$

$$\therefore x_0 = \frac{320\pi a^3}{63} \div \frac{32\pi a^2}{5} = \tfrac{5}{6}\tfrac{0}{3}a.$$

(5). Volume, $V = 2\pi\int_0^\pi\int_0^{a(1+\cos\theta)} rdrd\theta.r\sin\theta, = \frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3\sin\theta d\theta,$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3 d(1+\cos\theta), = \frac{8\pi a^3}{3}.$$

(6). The distance x_0 of this volume from the origin we find from

$$Vx_0 = 2\pi\int_0^\pi\int_0^{a(1-\cos\theta)} r^2 drd\theta\sin\theta r\sin\theta, = \frac{\pi a^4}{2}\int_0^\pi (1+\cos\theta)^4\sin^2\theta d\theta,$$

$$= 32\pi a^4\int_0^\pi \cos^{10}\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta d\theta, = \frac{64\pi a^4\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}{2\Gamma(7)}, = \tfrac{2}{2}\tfrac{1}{5}\pi^2 a^4;$$

$$\therefore x_0 = \frac{21\pi^2 a^4}{32} \div \frac{8\pi a^3}{3} = \tfrac{6}{2}\tfrac{3}{5}\tfrac{}{6}\pi a.$$

(7). The distance x_0 of the center of gravity of the arc of the curve from the origin is found by

$$sx_0 = 2\int_0^\pi r\sin\theta.2a\cos\tfrac{1}{2}\theta.d\theta, = 16a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta.\sin\tfrac{1}{2}\theta.d\theta,$$

$$= -32a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta d(\cos\tfrac{1}{2}\theta), = \frac{32a^2}{5}, \quad \therefore x_0 = \frac{32a^2}{5} \div 8a = \tfrac{4}{5}a;$$

and the distance x'_0 of the area of the curve from the origin is found by

$$Ax'_0 = \tfrac{2}{3}\int_0^{\pi} r^3\cos\theta d\theta, = \tfrac{2}{3}a^3\int_0^{\pi}(1+\cos\theta)^3\cos\theta d\theta, = \tfrac{16}{3}a^2\int_0^{\pi}(2\cos^8\tfrac{1}{2}\theta - \cos^6\tfrac{1}{2}\theta)d\theta,$$

$$= \tfrac{5}{4}\pi a^3. \qquad \therefore x'_0 = \tfrac{5}{4}\pi a^3 \div \frac{3\pi a^2}{2} = \tfrac{5}{6}a.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $r=a(1+\cos\theta)$, be the equation to the Cardioid.

(1). $s/2 = 2a\int_0^{\pi}\cos(\tfrac{1}{2}\theta)d\theta, = 4a. \quad \therefore s = 8a.$

(2). $A = 2a^2\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta, = \tfrac{1}{2}(3\pi a^2).$

(3). $C = 2\pi\rho$, where $\rho = r\sin\theta, = a\sin\theta(1+\cos\theta). \quad d\rho = 2a\cos^2\theta + a\cos\theta - a.$

$\therefore \cos\theta = \tfrac{1}{2}$ or -1. $\therefore$ for a maximum $\theta = 60^\circ$.

$\therefore C = 2\pi a(1+\cos\tfrac{1}{3}\pi)\sin\tfrac{1}{3}\pi, = \tfrac{1}{2}(3\sqrt{3}\pi a).$

(4). $K = 8\pi a^2\int_0^{\pi}\cos^3(\tfrac{1}{2}\theta)\sin\theta d\theta, = \tfrac{1}{5}(32\pi a^2).$

(5). $V = 2\pi\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta d\theta dr, = \frac{16\pi a^3}{3}\int_0^{\pi}\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta.$

$\therefore V = \tfrac{1}{3}(8\pi a^3).$

(6). $$x_0 = \frac{\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta d\theta dr}{\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta d\theta dr}, = \tfrac{3}{2}a\frac{\int_0^{\pi}\cos^8(\tfrac{1}{2}\theta)\cos\theta\sin\theta d\theta}{\int_0^{\pi}\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta}$$

$\therefore x_0 = \tfrac{4}{5}a.$

(7). $$g_0 = \frac{\int_0^{2\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta d\theta dr}{\int_0^{2\pi}\int_0^{a(1+\cos\theta)} r\, d\theta dr}, = \tfrac{4}{3}a\frac{\int_0^{2\pi}\cos^6(\tfrac{1}{2}\theta)\cos\theta d\theta}{\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta}$$

$\therefore g_0 = \tfrac{5}{6}a.$

Also solved by C. W. M. BLACK.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the law of the force, in order that the orbit may be a Cassinian Oval.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $r^4+2c^2r^2\cos 2\theta=m^4-c^4=a^4$ .....(1) be the equation to the oval.

Then $a^4u^4=2c^4u^2\cos 2\theta+1$, where $u=1/r$.

$$\frac{du}{d\theta}=\frac{c^2u\sin 2\theta}{c^2\cos 2\theta-a^4u^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\frac{d^2u}{d\theta^2}=\left\{\frac{(c^2\sin 2\theta\frac{du}{d\theta}+2c^2u\cos 2\theta)(c^2\cos 2\theta-a^4u^2 + (2c^2\sin 2\theta+2a^4u\frac{du}{d\theta})c^2u\sin 2\theta}{(c^2\cos 2\theta-a^4u^2)^2}\right\}$$

$$=\left\{\frac{3c^4r^8+10r^4c^4m^4-3r^4c^8-(m^4-c^4)^3-7r^4m^8+9r^8m^4-r^{12}}{r(a^4+2r^4)^3}\right\}.$$

$$F=\text{force}=h^2u^2(u+\frac{d^2u}{d\theta^2}) =\frac{h^2}{r^3}+\frac{h^2}{r^2}\cdot\frac{d^2u}{d\theta^2}.$$

$$\therefore F=\frac{h^2(7r^{12}+21r^8m^4+3r^4c^8-9c^4r^8-2c^4m^4r^4-m^8r^4)}{r^3(m^4-c^4+2r^4)^3}$$

$$=\frac{h^2\{7r^9+21m^4r^5-9c^4r^5+3rc^8-2c^4m^4r-m^8r\}}{(m^4-c^4+2r^4)^3}.$$

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a ball placed at a given latitude?

[No solution of this problem has been received. EDITOR].

42. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Find the time of vibration of a particle slightly displaced from the center of a solid cylinder in direction of the axis, the matter of the cylinder attracting according to the laws of nature.

Solution by the PROPOSER.

Consider the particle displaced by the amount x from the center of the cylinder. The matter attracting it will be a cylinder of length $2x$ at the opposite end of the cylinder. Call y the distance of any particle from the axis of cylinder and z the distance of particle from the end of cylinder with length $2x$ measured from the end towards the center.

The attraction of the cylinder upon the particle displaced from the center is

$$A=2\pi\int_0^R\int_0^{2x}\frac{y(a-x+z)dydz}{[(a-x+z)^2+y^2]^{\frac{3}{2}}}. \qquad A=2\pi\int_0^{2x}\left[dz-\frac{(a-x+z)dz}{[(a-x+z)^2+R^2]^{\frac{1}{2}}}\right].$$

$$=4\pi x-2\pi\sqrt{(a+x)^2+R^2}-2\pi\sqrt{(a-x)^2+R^2}$$

$$=4\pi x-2\pi\sqrt{(a^2+R^2)+(x^2+2ax)}-2\pi\sqrt{(a^2+R^2)+(x^2-2ax)}$$

$$=4\pi x-2\pi\{(a^2+R^2)^{\frac{1}{2}}+\tfrac{1}{2}\frac{x^2+2ax}{(a^2+R^2)^{\frac{1}{2}}}+\ldots\ldots\}$$

$$-2\pi\{(a^2+R^2)^{\frac{1}{2}}+\tfrac{1}{2}\frac{x^2-2ax}{(a^2+R^2)^{\frac{1}{2}}}\ldots\ldots\}$$

$$=4\pi x-4\pi(a^2+R^2)^{\frac{1}{2}}-2\pi\frac{x^2}{(a^2+R^2)^{\frac{1}{2}}}\ldots\ldots$$

Since the displacement is to be slight, we may neglect x^2 and all higher powers.

$\therefore (d^2x/dt^2)=4\pi x-4\pi(a^2+R^2)^{\frac{1}{2}}=ax-d$, for brevity.

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dt}. \quad \therefore v\frac{dz}{dx}=cx-d. \quad v^2=cx^2-2dx+k.$$

When the displacement is a maximum the particle is at rest. Call the amplitude α.

$$\text{Then } t=\int_0^{\alpha}\frac{dx}{\sqrt{cx^2-2dx-c\alpha^2+2d\alpha}}=\int_0^{\alpha}\frac{dx}{\sqrt{(2d\alpha-c\alpha^2)+(cx^2-2dx)}}$$

$$=\int_0^{\alpha}[(2d\alpha-c\alpha^2)+(cx^2-2dx)]^{-\frac{1}{2}}dx=\frac{\alpha}{\sqrt{2d\alpha-c\alpha^2}},$$

neglecting higher powers of x.

$$=\sqrt{\frac{\alpha}{8\pi\sqrt{a^2+R^2}}},\text{ neglecting the square of }\alpha.$$

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

47. Proposed by M. A. GRUBER, A. M., War Department, Washington D. C.

Find the first six sets of values in which the sum of two consecutive integral squares equals a square.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $\frac{1}{2}(x_n-1)$ and $\frac{1}{2}(x_n+1)$ be two consecutive integers, and y_n^2 the sum of their squares; then we must have

$$\tfrac{1}{4}(x_n-1)^2+\tfrac{1}{4}(x_n+1)^2=y_n^2, \text{ or } x_n^2-2y_n^2=-1 \ldots\ldots\ldots\ldots (1),$$

which may be written

$$(x_n-y_n\sqrt{2})(x_n+y_n\sqrt{2})=-1 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When $n=1$, we have

$$(x_1-y_1\sqrt{2})(x_1+y_1\sqrt{2})=-1 \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

also, raising (3) to the $(2n+1)$th power, we have

$$(x_1-y_1\sqrt{2})^{2n+1}(x_1+y_1\sqrt{2})^{2n+1}=-1 \ldots\ldots\ldots\ldots\ldots (4),$$

where n may be 0, 1, 2, 3, 4, etc.

Assuming $x_n-y_n\sqrt{2}=(x_1-y_1\sqrt{2})^{2n+1}$,

$x_n+y_n\sqrt{2}=(x_1+y_1\sqrt{2})^{2n+1}$, as we are at liberty to do, we find

$$x_n=[(x_1+y_1\sqrt{2})^{2n+1}+(x_1-y_1\sqrt{2})^{2n+1}]/2,$$

$$y_n=[(x_1+y_1\sqrt{2})^{2n+1}-(x_1-y_1\sqrt{2})^{2n+1}]/2\sqrt{2}.$$

It is easily seen that $x_1=1$ and $y_1=1$; therefore

$$x_n=[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}]/2, \quad y_n=[(\sqrt{2}+1)^{2n+1}+(\sqrt{2}--1)^{2n+1}]/2\sqrt{2},$$

and the required numbers are

$$\tfrac{1}{4}[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}-2] \text{ and } \tfrac{1}{4}[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}+2].$$

The operation of involution is very tedious except when n is a small number. When x_n and y_n are very large numbers we have from (1) very nearly

$x_n/y_n = \sqrt{2}$, and the values of x_n and y_n are the numerators and denominators of the odd convergents to $\sqrt{2}$ expanded as a continued fraction, after the first. The successive odd convergents are

1/1, 7/5, 41/29, 239/169, 1393/985, 8119/5741, etc.

The values of x_n and y_n are connected by the relations

$x_n = 6x_{n-1} - x_{n-2}$ .....(5), $y_n = 6y_{n-1} - y_{n-2}$ .....(6),

which afford an easy method of computing the successive sets of numbers required. When $n=1$ we easily find from the general formulas the first set, 3 and 4, and then from (5) the successive sets, which are

2nd set	20,	21,
3d "	119,	120,
4th "	696,	697,
5th "	4059,	4060,
6th "	23660,	23661.

See the *Mathematical Visitor*, Vol. I., No. 3, page 56, where the fifth and sixth sets are erroneously given as 4058, 4059 and 23657, 23658. The root of the sum of the squares of the sixth set should be 33461 instead of 33457. The 100 set is given on the same page; and also on page 122 where the numbers are found by Mr. K. S. Putnam by a different method.

These numbers solve the geometrical problem—"To find rational right-angled triangles whose legs are consecutive numbers."

II. Solution by A. H. BELL, Hillsboro, Illinois.

We have $x^2+(x+1)^2 = \square$, or $2x^2+2x+1 = \square$ .....(1).

Take $Ax^2 \pm Bx + C = \square = y^2$ .....(2).

(2) $\times A$, and add and subtract, etc.

$(Ax \pm B/2)^2 = Ay^2 + B^2/4 - AC = \square = t^2$ .....(3).

$\therefore t^2 - Ay^2 = B^2/4 - AC$ .....(A).

Also $x = (t \mp B/2)/A$ .....(B).

Let (A) reduce to $t^2 - Ay^2 = \pm D$ .....(4);

and a complete quotient $= (\sqrt{A} + M)/D$. Then the preceding convergent will $= t/y$ and will answer the $+$ or $-D$ as it is an odd or even number of fraction.

Also $v^2 - Au^2 = 1$ .....(5).

(4) $\times$ (5), and add and subtract $2Auvyt$, etc.

$(vt \pm Ayu)^2 - A(ut \pm vy)^2 = \pm D$, or $t_n^2 - Ay_n^2 = \pm D$ .....(C).

But in the problem (A) is $t^2-2y^2=-1$ .....(6).

Then we also have $t_n/y_n=(2vt_{n-1}-t_{n-2})/(2vy_{n-1}-y_{n-2})$ .....(7).

$\sqrt{2}$. Number of complete fractions=integer : 1, 2, etc.

Complete quotients=$(\sqrt{2}+0)/1$: $(\sqrt{2}+1)/1$, $(\sqrt{2}+1)/1$, etc.

Partial quotients=1 : 2, 2, etc.

Convergents=1/0, 1/1 : 3/2, 7/5, etc.

$\therefore$ $t/y=1/1$, 7/5, etc. $v/u=1/0$, 3/2. $2v_1=6$, $v_0=1$.

$\therefore$ (C) or (7) $t=1$, 7, 41, 239, 1393, 8119, 47321, etc.

(B) $x=0/2$, 3, 20, 119, 696, 4059, 23660, etc.

$x+1$ will give the six sets of values required.

NOTE. $2v=M$=magic M of Roberts and Robins.

III. Solution by the PROPOSER.

Take the formula for finding the sum of two integral squares equal to a square :

$$(2mp)^2+(m^2-n^2)^2=(m^2+n^2)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Then will the difference between $2mn$ and m^2-n^2 be 1. When $m^2-n^2>2mn$, we have $m^2-n^2-2mn=1$; whence $m=n\pm\sqrt{2n^2+1}$. When $2mn>m^2-n^2$, we have $2mn-(m^2-n^2)=1$, whence $m=n\pm\sqrt{2n^2-1}$. Substituting these values of m in (A), we obtain

$$[2n(n\pm\sqrt{2n^2\pm1})]^2+[2n(n\pm\sqrt{2n^2\pm1})\pm1]^2$$

$$=[2n(n\pm\sqrt{2n^2\pm1})+2n^2\pm1]^2 \quad \ldots\ldots\ldots (B).$$

It now remains to make $2n^2+1=\square$, and $2n^2-1=\square$. We find, by inspection, that the first value of n in $2n^2+1=\square$, is 0, and in $2n^2-1=\square$, is 1. Knowing these two values, we find the succeeding values from the formula, $n=2n_1+n_2$, in which n_1 is the last found known value of n, and n_2 the value just preceding. Whence $n=0$, 1, 2, 5, 12, 29, 70, 169, 408, 985, etc. *Zero* and the *even* numbers are the values of n in $2n^2+1=\square$, and the *odd* numbers in $2n^2-1=0$; the first two values of each series being known, the succeeding values can be found by the formula, $n=6n_1-n_2$.

It is also noticeable that the consecutive *odd number* values of n are the consecutive values of the root of the square that equals the sum of two consecutive integral squares. Substituting, now, the values of n in (B), we obtain, respectively, $0^2+1^2=1^2$, $4^2+3^2=5^2$, $20^2+21^2=29^2$, $120^2+119^2=169^2$, $696^2+697^2=985^2$, $4060^2+4059^2=5741^2$, $23660^2+23661^2=33461^2$, etc.

Or, from solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is taken as $n(n+1)/2$, the next in order, is terms of n is $[2n+1+3\sqrt{n(n+1)/2}]^2$. The difference of the roots of these two successive triangular square numbers is $2n+1+2\sqrt{n(n+1)/2}$. The sum of the roots is $2n+1+4\sqrt{n(n+1)/2}$, which, when $n(n+1)/2=\square$, equals the sum of the two *consecutive integral* numbers, $n+2\sqrt{n(n+1)/2}$ and $n+1+2\sqrt{n(n+1)/2}$.

But $[n+2\sqrt{n(n+1)/2}]^2+[n+1+2\sqrt{n(n+1)/2}]^2$

$$=6n^2+6n+1+(8n+4)\sqrt{n(n+1)/2}=[2n+1+2\sqrt{n(n+1)/2}]^2.$$

We here have a general formula in which the sum of the squares of two consecutive integers equals a square. To obtain integral numerical results, we assign the successive values of n in $n(n+1)/2=\square$, 1, 8, 49, 288, 1681, 9800, etc. Whence we have $3^2+4^2=5^2$; $20^2+21^2=29^2$; $119^2+120^2=169^2$; $696^2+697^2=985^2$; $4059^2+4060^2=5741^2$; $23660^2+23661^2=33461^2$, etc.

IV. Solution by Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Let x=one number and $x+1$=the other; then x^2+x^2+2x+1 will be the sum of two consecutive squares. Then $2x^2+2x+1=\square=$(say) $(mx-1)^2$, from which we readily obtain $x=2(m+1)/(m^2-2)$. It is readily seen that x is integral when $m=2$. Then we have 2/1, 10/7, 58/41, etc., for values of m which give integral values of x, viz., 3, 119, 4059, etc. The other series which makes x integral is 3/2, 17/12, 99/70, etc., and x=20, 696, 23660, etc. The six values of x, therefore, are 3, 20, 119, 696, 4059, 23660, and of $x+1$, 4, 21, 120, 697, 4060, 23661, and the squares of these values are *probably* the squares required. I say "probably," because it cannot be mathematically determined that some other method of solution will not give other results that show that there are other values less than 23660 besides those I have given.

[The following is my formula for obtaining integral values of a fraction whose denominator is p^2-2, which I assume in this solution. I have never seen the formula in print and do not know how generally it is known.

If r/s is of such a value of p as will give an integral result, then $(3r+4s)/(2r+3s)$ is *another* value of p that gives an integral result, and so on *ad infinitum*. If the numerator is even, there will be two different series of values of p, the initial term in one being $2/1$, and in the other $3/2$; if the numerator is odd, the series beginning with $3/2$ will give integral results. By means of this formula an infinite number (mathematically speaking) of integral values of x may be obtained in the equation $2x^2+ax+b^2=\square$, in terms of a and b; and in the equation $2x^2+2ax+b^2$, two series (infinite) of integral values in terms of a and b may be obtained. In both cases, however, the numbers increase in value very rapidly.]

V. Solution by A. H. HOLMES, Brunswick, Maine.

$x^2+(x+1)^2=\square$ or $2x^2+2x+1=\square$. Let $x=y+p$.

$\therefore 2y^2+(4p+2)y+2p^2+2p+1=\square$, from which we find the law of the series to be: $b=1+3a+2\sqrt{2a^2+2a+1}$. Let $a=3$ and we find $b=20$. Then by the same law, $c=119$, $d=696$, $e=4059$, and $f=23660$. Therefore, we have for the first six sets of values: (3 and 4), (20 and 21), (119, 120), (696 and 697), (4059 and 4060), and (23660 and 23661).

VI. Solution by H. C. WILKES, Skull Run, West Virginia.

We have $x^2+(x+1)^2=y^2=4n+1$, then $x(x+1)/2=n$. Substituting this value for n in $4n+1=y^2$ we have $x^2+x=(y^2-1)/2$. Putting $x+(x+1)=t$ or $x=(t-1)/2$, we obtain $t^2-2y^2=-1$. Since $t=7$, $y=5$ satisfy this equation, the first values of $x+(x+1)$ and y will be 7 and 5.

$\therefore 3^2+4^2=5^2$. From inspection of solution II, Problem 36, Vol. III, page 81, we find a formula for obtaining the succeeding values of $x+(x+1)$ and y.

$x++(x1)$.	y.	
$6\times7-1=41$,	$6\times5-1=29$,	$20^2+21^2=29^2$,
$6\times41-7=239$,	$6\times29-5=169$,	$119^2+120^2=169^2$,
$6\times239-41=1393$,	$6\times169-29=985$,	$696^2+697^2=985^2$,
$6\times1393-239=8119$,	$6\times985-169=5741$,	$4059^2+4060^2=5741^2$,
$6\times8119-2392=47321$.	$6\times5741-985=33461$.	$23660^2+23661^2=33461^2$.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

48. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

If any positive integral number N be divided by another positive integral number D, leaving a remainder 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $N=nD+1$, then

$$(nD+1)^m=n^mD^m+mn^{m-1}D^{m-1}+\frac{m(m-1)}{2}n^{m-2}D^{m-2}+$$

$$\ldots\ldots+\frac{m(m-1)}{2}n^2D^2+mnD+1,$$

$$=D[n^mD^{m-1}+mn^{m-1}D^{m-2}+\frac{m(m-1)}{2}n^{m-2}D^{m-3}+\ldots\ldots+\frac{m(m-1)}{2}n^2D+mn]+1,$$

which proves the proposition.

Solved in a similar manner by *M. A. GRUBER* and *G. B. M. ZERR*.

II. Solution by J. C. CORBIN, Pine Bluff, Arkansas; P. S. BERG, Larimore, North Dakota; E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont; A. P. READ, A. M., Clarence, Missouri; and O. S. WESTCOTT, Principal North Chicago High School, Chicago.

Put $N=nD+1$, then it is evident that if $N=nD+1$ be raised to any positive integral power, the last term will be 1 and every other term will contain D as a factor; hence if this power be divided by D the remainder will be 1.

Also solved in a similar way by *A. H. BELL, JOSIAH H. DRUMMOND, ARTEMAS MARTIN* and *J. SCHEFFER*.

III. Solution by J. O. MAHONEY, B. E., M. S., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

If $a\equiv a'$, $b\equiv b'$, $c\equiv c'$, $d\equiv d'$, etc., mod(D),

then $abcd.....\equiv a'b'c'd'.....mod(D)$.

Let $a=b=c=d$, etc.,$=N$, and $a'=b'=c'=d'$, etc.,$=1$, then $N^t\equiv 1$ mod(D).

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

No solution of this problem has been received.

40. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

If every point of an ellipse be joined with every other point, what is the average length of the chords thus drawn?

Solution by the PROPOSER.

Let $a\cos\theta$ and $(b/a)\sin\theta$ be the coördinates of one point, and $a\cos\phi$ and $(b/a)\sin\phi$ those of another.

The length of the chord joining them is

$$K=[a^2(\cos\phi-\cos\theta)^2+\frac{b^2}{a^2}(\sin\phi-\sin\theta)^2]^{\frac{1}{2}}.$$

Let s_1 and s_2=lengths of elliptic arcs from point $(a, 0)$ to points $(a\cos\theta, \frac{a}{b}\sin\theta)$ and $(a\cos\phi, \frac{a}{b}\sin\theta)$ respectively, and let S=whole distance around the ellipse.

Then $\frac{ds_1}{d\theta}=a(1-e^2\cos^2\theta)^{\frac{1}{2}}$ and $\frac{ds_2}{d\phi}=a(1-e^2\cos^2\phi)^{\frac{1}{2}}$.

Then the required average is

$$A=\frac{H}{S^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}Kds_1\,ds_2=\frac{4a^2}{S^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}[a^2(\cos\phi-\cos\theta)^2+\frac{b^2}{a^2}(\sin\phi-\sin\theta)^2]^{\frac{1}{2}}\times$$

$$(1-e^2\cos^2\theta)^{\frac{1}{2}}\,(1-e^2\cos^2\phi)^{\frac{1}{2}}\,d\theta d\phi.$$

This equation cannot be integrated in general terms.

Solved in the same manner by *G. B. M. ZERR*,

41. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A line is drawn at random across the chord and given arc of a circular segment. Find the mean area of the divisions.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A=area of given segment, A_1, A_2 mean areas of the two divisions.

$\therefore A_1+A_2=A$.

But, since the line is a random line, $A_1=A_2$.

$\therefore A_1=A_2=\frac{1}{2}A$.

Also solved by *HENRY HEATON*.

42. Proposed by CHARLES E. MYERS, Canton, Ohio.

A attends church 4 Sundays out of 5; B, 5 Sundays out of 6; and C, 6 Sundays out of 7. What is the probability of an event that A and B will be at church and C will not?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

The chance that A attends church=$\frac{4}{5}$.

The chance that B attends church=$\frac{5}{6}$.

The chance that C attends church=$\frac{6}{7}$.

The chance that A is not at church=$\frac{1}{5}$.

The chance that B is not at church=$\frac{1}{6}$.

The chance that C is not at church=$\frac{1}{7}$.

The chance that A and B attend and C not=$p_1=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{1}{7}=\frac{2}{21}$.

The chance that A and C attend and B not=$p_2=\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{6}=\frac{4}{35}$.

The chance that B and C attend and A not=$p_3=\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{1}{5}=\frac{1}{7}$.

The chance that A attends and B and C not=$p_4=\frac{4}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}=\frac{4}{210}$.

The chance that B attends and A and C not=$p_5=\frac{1}{5}\cdot\frac{5}{6}\cdot\frac{1}{7}=\frac{1}{42}$.

The chance that C attends and A and B not=$p_6=\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{6}{7}=\frac{1}{35}$.

The chance that A, B and C attend=$p_7=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}=\frac{4}{7}$.

The chance that A, B and C do not attend=$p_8=\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}=\frac{1}{210}$.

p_1=probability required.

Also $p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8=1$.

Also solved by *HENRY HEATON*.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. M. PRIEST, Mona House, St. Louis, Missouri.

Suppose two cylindrical iron shafts, each 6 inches in diameter and respectively, 20 and 40 feet in height, are both standing perpendicular at the sea level. They start to fall in still air, how long will it require each one to fall to a horizontal position?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Neglecting the atmosphere and supposing the cylinder to revolve about a diameter in the base, we get, if l is the length of an equivalent pendulum, from works on Mechanics the formula for the time of vibration of a pendulum,

$$t=\sqrt{\frac{l}{g}}\int_{\theta}^{a}\frac{d\theta}{\sqrt{\sin^2\tfrac{1}{2}a-\sin^2\tfrac{1}{2}\theta}}.$$

In this problem a is $180^\circ=\pi$, $\theta=90^\circ=\frac{1}{2}\pi$.

$$\therefore t=\sqrt{\frac{l}{g}}\int_{\frac{1}{2}\pi}^{\pi}\frac{d\theta}{\cos\frac{1}{2}\theta}=\left[2\sqrt{\frac{l}{g}}\log.\left\{\frac{\tan(\frac{1}{4}\pi+\frac{1}{4}a)}{\tan(\frac{1}{4}\pi+\frac{1}{4}\theta)}\right\}\right]_{\theta=\frac{1}{2}\pi}^{a=\pi}$$

$\therefore t=\infty$, which proves that in a perfectly vertical position they will not fall unless moved slightly from this position. Let $a=\pi-\delta$ where δ is very small.

$$\therefore t=2\sqrt{\frac{l}{g}}\log.\left\{\frac{\cot\frac{1}{4}\delta}{\tan 3\pi/8}\right\}.$$

Let $\delta=1'$, $l=$length of cylinder, $b=$radius of base.
$\therefore l=(3b^2+4l^2)/6l=13.3349$ feet for first cylinder.
$l=26.66745$ feet for second cylinder.
$\therefore t=2(.644328)(3.153498)=4.0638$ seconds for first cylinder.
$t=2(.911177)(3.153498)=5.7468$ seconds for second cylinder.

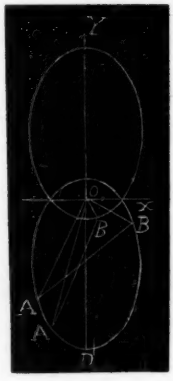
41. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

A straight inflexible bar of uniform weight and thickness, length m is suspended at the two ends by a string without weight, length $l>m$ passing freely over a peg driven in a perpendicular wall. Describe and analyze the curve traced on the wall by the ends of the hanging bar.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let O be the peg, AB the rod. Let $OB=r$, $AO=r'$, $AB=m$, $\angle XOB=\theta$, $\angle XOA=\phi$.

Now in equilibrium OD always passes through the mid-point of AB.



Then $r+r'=1$ .....(1).

$m^2=r^2+r'^2-2rr'\cos(\phi-\theta)$ ....(2).

$r\cos\theta=r'\cos\phi$ .....(3).

(3) is obtained from the two triangles OAC, OBC.
(1) in (2) and (3) gives

$m^2=r^2+(1-r)^2-2r(l-r)\cos(\phi-\theta)$ ..... .(4).

$r\cos\theta=(1-r)\cos\phi$ .....(5).

(5) in (4) gives $l^2-m^2+2r^2\sin^2\theta-2rl=2r\sin\theta\sqrt{l^2-2rl+r^2\sin^2\theta}$.

$\therefore 4r^2(l^2-m^2\sin^2\theta)-4rl(l^2-m^2)+(l^2-m^2)^2=0$.

$$\therefore r=\frac{l^2-m^2}{2(l\pm m\sin\theta)}=\frac{l(1-e^2)}{2(1\pm e\sin\theta)}, \text{ where } e=m/l.$$

This equation represents two equal ellipses with eccentricity$=m/l$, major axis$=l$, minor axis$=\sqrt{l^2-m^2}$, and O is one focus of each ellipse.

[The above solves the problem—"An ellipse confined to one vertical plane is suspended from a fixed point in space, coincident with a movable point on its circumference. Describe the curve marked out by foci." EDITOR].

EDITORIALS.

With January, 1897, the Chicago *Open Court* celebrates its decennial anniversary and now appears in the form of a monthly instead of a weekly.

Plane and Solid Analytical Geometry, by Frederick H. Bailey, A. M., and Frederick S. Woods, Ph. D., Assistant Professors of Mathematics in Massachusetts Institute of Technology, is announced as ready in March by Ginn and Company.

President H. H. Seerley, of the State Normal School of Cedar Falls, Iowa, has just ordered a complete set of the MONTHLY for the library. We only have a few more complete sets. Who wants them?

We are in correspondence with several excellent mathematicians who are desirous of securing better positions for next year. If any of our readers know of such positions which are vacant or likely to become vacant at the end of this school year, we shall be pleased to refer them to these gentlemen.

With this number begins the fourth volume of the MONTHLY. No pains will be spared on the part of the Editors to make this volume better than any of the three previous ones, and in this effort they earnestly solicit the continued aid of all former contributors and subscribers. This number is sent to all our old subscribers, with bill enclosed, and anyone who may wish to discontinue should return this copy with his name written on the wrapper.

BOOKS AND PERIODICALS.

Elements of Analytical Geometry of Two Dimensions. The Fourteenth Edition. By Briot and Bouquet. Translated and Edited by James Harrington Boyd, Instructor in Mathematics in The University of Chicago. 8vo. Cloth, 582 pages. Introduction Price, $2. Chicago: Werner School Book Co.

This celebrated work so long known to mathematicians familiar with the French language, is now put in English dress, and is, therefore, at the service of American students. Comments on the material and the method of this work are unnecessary.

The work is divided into four books. Book I contains four chapters: Chapter I, Concerning Coördinates; Chapter II, Examples—The Circle, the Ellipse, the Hyperbola, the Parabola, Cissoid of Diocles, etc.; Chapter III, Concerning Homogenity; Chapter IV, Transformation of Coördinates. Book II contains three chapters: Chapter I, Straight Line; Chapter II, the Circle; Chapter III, the Geometrical Loci. Book III contains twelve chapters: Chapter I, Construction of Curves of the Second Degree; Chapter II, Center, Diameter, and Axes of Curves of the Second Degree; Chapter III, Reduction of the Equation of the Second Degree; Chapter IV, the Ellipse; Chapter V, the Hyperbola; Chapter VI, Concerning the Parabola; Chapter VII, Foci and Directrices; Chapter VIII, the Conic Sections; Chapter IX, the Determination of the Conic Sections; Chapter X, Theory of Poles and Polars; Chapter XI, General Properties of Conic Sections; Chapter XII, Secants Common to Two Conics. Book IV contains seven chapters: Chapter I, the Construction of Curves in Rectilinear Coördinates; Chapter II, Convexity and Concavity; Chapter III, Asymptotes; Chapter IV, Construction of Curves in Polar Coördinates; Chapter V, Concerning Similitude; Chapter VI, Graphic Solutions of Equations; Chapter VII, Notions Concerning Unicursal Curves.

From the table of contents it is seen that a leading feature of the work is its scope. It treats all the important methods invented by geometers, and includes some of the most beautiful discoveries of ancient and modern times. All subjects are treated in a practical way and illustrated by the applications of the theories to numerous problems. The book is beautiful as well as profound. The typographical and mechanical execution of the work is a credit to American text-book making. I very heartily commend this work to the careful consideration of teachers of Analytical Geometry and mathematical students desiring a good work on the subject. B. F. F.

The Outlines of Quaternions. By Lieutenant-Colonel H. W. L. Hime. 188 pages. Price, $3. Longmans, Green & Co. 1894. London and New York.

The first chapters deal with the properties of vectors. In the remaining pages we are introduced to quaternions proper,—their various forms and properties. The last chapter treats of the applications of quaternions to trigonometry, the triangle, the circle, conic sections, and other curves, the plane, tetrahedron, sphere and cone. These geometric applications show in some measure the usefulness of quaternions and give freshness and interest to the book. There is no preface. The addition of some exercises for solution would have added to the practical character of the work for class use. J. M. C.

Plane Surveging. By William G. Raymond, C. E., Member American Society of Civil Engineers; Professor of Geodesy, Road Engineering, and Topographical Drawing, in the Rensselaer Polytechnic Institute, Troy, New York. 8vo. Cloth, 486 pages (including tables). Price, $3. Chicago: American Book Co.

Some of the valuable features of this work are the detailed description of the use of instruments, accompanied by excellent illustrations and diagrams of the instruments themselves; the clear and comprehensible presentation of the subject matter of the work; and the fine form in which it appears for public favor. In its pages may be found treated plane table work and the use of the slide rule, planimeter and stadia measurements. Full tables and numerous examples of work in the way, both of underground and general topography are also given. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

The *Review of Reviews* for February makes "A Plea for the Protection of Useful Men" from bores and "societies," and all well-meaning people who bother the life out of public men by letters and calls on the pretext of seeking assistance in some worthy undertaking. The editor of the *Review* publishes letters on this subject from the late Gen. Francis A. Walker, written only a few weeks before his death. In one of these letters General Walker wrote, "I am not well, and neither callers nor correspondents have any mercy.

B. F. F.

The Cosmopolitan. An Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1. per year in advance. Single number, 10 cents. The Cosmopolitan Co., Irvington-on-the-Hudson, New York.

The January number of the *Cosmopolitan* not only keeps up the usual literary excellence, artistic merit, and widest interests of that magazine, but also adds new features to its field of usefulness. The February number will contain the second part of Conan Doyle's new story.

During the year 1896, the *Cosmopolitan* reached the largest clientèle of intelligent, thoughtful readers possessed by any periodical in the world. The smallest issue of the year was 300,000 copies. B. F. F.

The Arena. A Monthly Magazine. Price, $3. Single number, 25 cents. Boston: Arena Publishing Co.

The *Arena* is the organ or mouth-piece of no one party, faction, or creed. It is unmortgaged and unbribed—a *free lance*, an *open arena*—wherein all honest and properly expressed and authoritative opinions, having in view the betterment of human conditions and human life, may be expressed. The best writers and authorities on leading questions contribute to its pages. Among the leading articles in the January number are the following: The Religion of Burns' Poems, by Rev. Andrew W. Cross; A Court of Medicine and Surgery, by A. B. Choate; Finance and Currency, by Gen. Heman Haupt; Daniel Webster's School Days, by Forest Prescott Hall; England's Hand in Turkish Massacres; etc.

B. F. F.